More Ado About Nothing

Abstract

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In this paper questions about vacuum fluctuations in local measurements, and the correlations between such fluctuations are discussed. It is shown that maximal correlations always exist between suitably chosen local projection operators associated with space-like separated regions of space-time however far apart these regions are placed. The connection of this result with the well-known ~~bound~~ Fredenhagen bound stating exponential decay of correlations with distance is explained and the relevance of the discussion to the question 'what do particle detectors detect?' is addressed.

Equation I

We consider the situation at a fixed time.

More Ado About Nothing

1. Introduction

In relativistic quantum field theory the vacuum behaves very differently from a global and a local point of view. Globally the vacuum is the state of lowest energy, identified by the zero eigenvalue for particle and anti-particle number operators. So it is a state with no particles in it. But locally it is seething with activity. Charge densities and other local observables exhibit fluctuations and correlations, which produce observable phenomena, such as very accurately predicted contributions to the magnetic moment exhibited by an electron in a magnetic field. In order to understand why the relativistic vacuum behaves in such a remarkable way let us begin by contrasting the situation with non-relativistic quantum field theory. If we quantize the Schrödinger field we obtain the second-quantized version of the N-particle Schrödinger equation.

The number operator $N = \int \psi^*(x)\,\psi(x)\,d^3x$ has eigenvalues $0, 1, 2, \ldots$ associated with definite numbers of particles located somewhere in space. But we can introduce operators $N_V = \int_V \psi^*(x)\,\psi(x)\,d^3x$, associated with the number of particles in a spatial volume V. For two disjoint volumes V and V', N_V and $N_{V'}$ commute, while both commute with N. So if we cover the whole of space with a collection of disjoint volumes V_i, then

we can set up a state of the field associating a definite number N_i of particles with the volume V_i so that ΣN_i sums to the total number of particles N in that particular state.
The vacuum is the state with $N=0$ and hence also all the N_i for any disjoint covering of the whole of space must also be zero. In other words it makes sense to say that the global vacuum is also a local vacuum.
This nice connection between the global and the local vacuum is what breaks down in relativistic field theories: Attempts to define local number operators for particles and antiparticles $N_V^{\pm}$ corresponding to N_V in the above discussion produce operators which fail to commute for disjoint volumes and don't commute with the total number operators $N^{\pm}$.
So it is no longer possible to have a state of the field which has simultaneously sharp values for the global (total) number operators and also for the local number operators.
In particular the global vacuum, where $N^{\pm}=0$, can no longer be identified as the state where the local number operators have vanishing eigenvalues.
The standard gloss (1) put on this state of affairs in the physics literature is that in relativistic quantum field theory, virtual pairs of particles and

antiparticles can be created locally in the field, and this possibility of ~~local pair creation~~ is what spoils the ~~possibility of~~ sharp values for the local operators N_V^{+}.

But this type of interpretation can be potentially misleading. It suggests that there are localized particle states which must in general be superimposed to get the global particle states. I want to argue in this paper for a different sort of interpretation, viz. that in the relativistic theories ~~that~~ there are no such thing as localized particle states, that the whole concept of a particle state in relativistic quantum field theory is associated with global aspects of the theory.

There are two lines of argument here

(1) Particle states arise in quantum field theory via asymptotic scattering states. Such states are associated with definite momentum, but no precise localization

(2) Attempts to define an invariant, i.e. objective, position operator for relativistic particles is doomed to failure. Particles if they can be localized at all, can only be localized in one Lorentz frame. The boosted states are no longer localized. This is the essential reason for the superluminal dispersion of localized states discovered by Hegerfeldt[2].

There is no causality violation here, because the ~~the~~ states are not really localized at all, when account is taken of description from different Lorentz frames.(3)

In order to assess the status of the local vacuum in relativistic quantum field theory, and its relation to global 'particle' states, I shall now pursue the investigation in the framework of algebraic quantum field theory.(4,5)(2,5) Here one associates ~~a~~ an algebra of local observables $R(O)$ with every bounded open region O in space-time.

(Newman)

In addition we assume a global vacuum state Ω and a Hilbert space H in terms of which we can represent the action of a space-time translation $\underline{a}$ on the algebra $R(O)$ in the form

$$R(O+a) = U(\underline{a})\, R(O)\, U^*(\underline{a})$$

Here U is a unitary operator acting on H and $O+a$ is the image of O under the translation $\underline{a}$.

Ω is assumed to be the unique state which is invariant under any translation operator $U(\underline{a})$.

For time-like translations we can exponentiate $U(\underline{a})$ to obtain a Hamiltonian operator which is assumed to be non-negative, i.e. the energy spectrum of the field has no negative elements.

In addition it is customary to introduce the ~~quasi-local~~ global algebra R, defined as the

~~The Union of all the local algebras we shall refer~~

smallest von Neumann algebra containing all the local algebras, and we assume that the representation of R_{univ} is irreducible and generated by the translates of $R(O)$ for any bounded region O.

There are two further important properties of the net of local algebras $\{R(O)\}$, which we shall assume:

Isotony: For any two bounded open sets O_1 and O_2, $O_1 \subseteq O_2 \Rightarrow R(O_1) \subseteq R(O_2)$

Locality: For all bounded open sets O_1 and O_2, if O_1 and O_2 are space-like related (i.e. every point in O_1 is spacelike related to every point in O_2) then every operator in $R(O_1)$ commutes with every operator in $R(O_2)$.

From these postulates we can derive one of the most famous results in axiomatic quantum field theory, the Reeh-Schlieder theorem[6], which, as we shall see, is the key to understanding the nature of the vacuum in relativistic quantum field theory.

2. The Reeh-Schlieder Theorem and Its Implications

We first explain what is meant by the claim that Ω is cyclic for $R(O)$ with respect to the Hilbert space H. This just means that $\{A\Omega : A \in R(O)\}$ is dense in H, or in other words acting on Ω with arbitrary elements of $R(O)$ can approximate as closely as we like any vector in H.

The Reeh-Schlieder theorem just says:
Let O be any bounded open set. Then Ω is cyclic for $R(O)$.

Why is this result so surprising, even paradoxical?
In pre-axiomatic discussions of quantum field theory, the Hilbert space H was regarded as being scaffolded by eigenstates of particle number. These eigenstates were themselves all generated from the vacuum state by suitable creation operators. So in other words any vector in H could be built up by superimposing the result of suitable operators acting on Ω.
In the language of algebraic quantum field theory, this makes it reasonable to assume that acting on Ω with suitable elements of the ~~quasilocal~~ global algebra R, we might expect at least to approximate as near as we like any ~~state~~ vector in H.
In other words we would quite expect Ω to be cyclic for R with respect to H.

But the Reeh-Schlieder result is much stronger than that: It claims that Ω is cyclic for $R(O)$, where O is any bounded open set in spacetime. So O might just be the neighbourhood of some particular point in space-time. But then, how could acting with the elements of such an $R(O)$ on Ω, approximate an arbitrary state of the field, eg particular one which looks quite unlike the vacuum in some distant, spacelike separated neighbourhood O', without involving gross violations of locality?

Before discussing the significance of this result and the resolution of the apparent paradox, I first want to draw attention to an important corollary of the Reeh-Schlieder theorem: Ω is not only cyclic for $R(O)$, but is also a <u>separating vector</u> for $R(O)$. What this means is that if $A_O \in R(O)$ then $A_O\Omega = 0 \Rightarrow A_O = 0$.
In other words if two elements A_1 and A_2 of a local algebra yield the same vector when acting on Ω, they must be one and the same operator, so Ω is sufficiently 'rich' in structure to discriminate the action of any two distinct elements of any local algebra.

How are we to interpret the Reeh-Schlieder theorem? We begin by making some remarks about the nature of the operators occurring in $R(O)$. First of all there are projection operators, which we shall designate generically by P. These operators have eigenvalues 0 or 1, and we shall

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associate them with the result of performing selective measurement operations by means of experimental procedures localized in O.
by selective we mean the operation of forming subensembles which are homogeneous in respect of the outcome of performing the measurement operation in question.
So in the state Ω if we perform the operation associated with the projector P the state after the selection procedure is just $\frac{P\Omega}{\|P\Omega\|}$. This is just the familiar projection postulate.
Now in a von Neumann algebra[3] all the operators can be built up by linear combination and limit operations from the projection operators.
This does not mean that if $A \in R(O)$ then $P_{A\Omega}$, i.e. the projector onto the state $A\Omega$ is itself a member of $R(O)$. Far from it. Consider for example that $I \in R(O)$, and then ask, is $P_\Omega \in R(O)$? The answer is no since $P_\Omega \in R(O)$ $\Rightarrow 1-P_\Omega \in R(O) \Rightarrow (1-P_\Omega)\Omega = 0$, $\Rightarrow P_\Omega = 1$ which is impossible since P_Ω is a one-dimensional projector, i.e. its range is the one-dimensional ray associated with the vector Ω. The fact that $P_\Omega \notin R(O)$ ~~in fact it is~~ means just that it is never a local question to ask 'are we in the state Ω?'. This could only be answered by surveying the whole of space-time but no local procedure can do this. Similarly if Ψ is an N-particle state it will be orthogonal to Ω so the corresponding projector P_Ψ will satisfy $P_\Psi\Omega = 0$

But this is infeasible if $P_4 \in R(O)$, since by the Reeh-Schlieder corollary it would imply $P_4 = 0$.
So again it is not a local question to ask 'are we in an N-particle state?' Particle states of which the vacuum is a special case are essentially non-local objects.
We can actually strengthen the results of the above discussion to obtain the following

Theorem 1: If $P \in R(O)$ then P is an infinite-dimensional projector.

Proof: This follows directly from the result of Driessler[7] which states that the quasi-local algebra associated with an unbounded wedge of space time, ~~associated~~ identified ~~with the exterior of the light cone erected at any point in space-time~~ is a type III factor.
Now any bounded region is internal to some wedge so by isotony $R(O)$ is a subalgebra of some wedge algebra. So the projectors in $R(O)$ are identified with some of the projectors in the wedge algebra. But in a type III factor all the projectors are infinite-dimensional. So all the projectors in $R(O)$ are infinite-dimensional.
Intuitively what is ~~going on~~ happening here is that local measurements can never establish what is going on outside O, ~~so in respect of all observables lying outside O~~ (at space-like separation), so in respect of all

algebras associated with regions space-like situated with respect to O the local projectors in R(O) behave like the identity.. This is what prevents it from being one-dimensional or indeed finite-dimensional.

Let us now consider some further properties of the ~~local~~ projectors which are members of R(O).

Define $p = \mathrm{Prob}^{\Omega}(P \in R(O) = 1)$

So p is the probability that in the vacuum state ~~the~~ a local measurement procedure with specified outcome will produce that outcome.

Then $p = \|P\Omega\|^2$ so $p = 0 \Rightarrow P\Omega = 0$ $\Rightarrow P = 0$.

So if $P \neq 0$ we conclude that $p \neq 0$.

~~In other words~~ Since we have

Theorem 2[8] Any possible outcome of any possible measurement procedure will occur with non-vanishing probability in the vacuum.

In other words if we place a detector in the vacuum designed to respond to any arbitrary excitation (state) of the field, there is a finite probability that it will so respond.

Theorem 2 shows just just how 'rich' a state the vacuum really is. In the long run anything that is possible will happen in the vacuum.

Another way of understanding theorem 2 is that the range of any non-trivial projector

is never orthogonal to the vacuum, i.e. it is never parallel to any particle state. But equally we can deduce that the projector $I - P$ is never orthogonal to the vacuum i.e. that P is never parallel to the vacuum. Taken together these two results show that non-trivial local measurements never produce particle states. A greater indication of how far removed from any local concept is that of a particle state in relativistic quantum field theory.

So far we have shed no light on the mechanism involved in the Reeh-Schlieder theorem. To do this we first must consider the question of vacuum correlations. Suppose we had a projector $P_2 \in R(O_2)$ and another projector $P_1 \in R(O_1)$, ~~if the conditional probability $Prob^{\Omega}($~~ where O_1 and O_2 were spacelike separated. Then if the conditional probability $Prob^{\Omega}(P_2 = 1 / P_1 = 1)$ were equal to one, then measuring P_1 and selecting a positive outcome would force Ω into a state which was not only in the range of P_1 but also in the range of P_2, so operations performed in O_1 could produce changes in the state, as evaluated in O_2, by exploiting the perfect correlations between P_1 and P_2.

If this could be achieved whatever the choice of P_2 and remembering that arbitrary operators in $R(O_2)$ can be built up from projectors, this would give us a clue

as to how arbitrary operations in O_2 ~~can~~ could be ~~controlled by~~ effected generated by operations localized in O_1.

As a warming-up example let us consider a baby version of the Reeh-Schlieder theorem which applies to two spin-$\frac{1}{2}$ particles in the singlet state of their total spin.

Our claim is that operations localized in one factor space can change the state to an arbitrary vector in $H_1 \otimes H_2$ where H_1 and H_2 are the two-dimensional Hilbert spaces describing the individual spins.

Following Licht[9] we want to distinguish clearly two senses of the term 'operation'. Firstly, there are physical operations such as making measurements, selecting subensembles according to the outcomes of measurement and mixing ensembles with probabilistic weights, and secondly there are the mathematical operations of producing superpositions of states by taking linear combinations of pure states produced by appropriate selective measurement procedures. These superpositions are of course quite different from the physical mixed states ~~prepared by~~ whose preparation we have listed as a physical operation.

In order to produce arbitrary states in $H_1 \otimes H_2$ we have to invoke both physical and mathematical operations.

Thus let us write the singlet state in ~~the~~ form

$$\cancel{|\Psi_{singlet}\rangle = \frac{1}{\sqrt{2}}\left(|\alpha(1)\rangle|\beta(2)\rangle - |\beta(1)\rangle|\alpha(2)\rangle\right)}$$

~~where~~ $\cancel{|\alpha(i)\rangle, |\beta(i)\rangle}$ ~~are the eigenstates of the Pauli spin operator~~ $\cancel{\sigma_{1z}}$ ~~for particle one with~~

$$|\Psi_{\text{singlet}}\rangle = \frac{1}{\sqrt{2}}\Big(|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle - |\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle\Big) \quad (1)$$

where $|\sigma_{1z}=+1\rangle$, $|\sigma_{1z}=-1\rangle$ are the eigenstates of the Pauli spin operator σ_{1z} for particle one with eigenvalues +1 and −1 respectively. Similarly $|\sigma_{2z}=+1\rangle$, $|\sigma_{2z}=-1\rangle$ are spin states referring to particle two.

Now if we measure σ_{1z} and select the outcome +1 then we have physically produced the state $|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$. Similarly by measuring σ_{1z} and selecting the outcome −1, we can produce the state $|\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$.

But referring to the state $|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$ we can by a further physical procedure produce the state $|\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=-1\rangle$. We just have to apply a magnetic field along the y-axis to particle one and allow the spin to precess for one half the Larmor period. Similarly by the physical process of Larmor precession we produce the state $|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=+1\rangle$ from the state $|\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$. So by means of physical operations on particle one and exploiting the mirror-image correlations built into $|\Psi_{\text{singlet}}\rangle$ we can produce the states $|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=+1\rangle$, $|\sigma_{1z}=+1\rangle \otimes |\sigma_{2z}=-1\rangle$, $|\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=+1\rangle$ and $|\sigma_{1z}=-1\rangle \otimes |\sigma_{2z}=-1\rangle$ for the joint

*Insertion: But all the operations we have described can be represented by the algebra of operators on H_1, # So, if we denote the (von Neumann) algebra of operators on H_1 by R_1 (and similarly the algebra of operators on H_2 by R_2) then the baby ~~Reeh-Schlieder~~ R-S theorem can be formulated as

$$\forall \phi \in H_1 \otimes H_2, \ \exists A_1 \in R_1 \ \text{s.t.} \ |\phi\rangle = A_1 |\psi_{\text{singlet}}\rangle$$

and similarly of course.

$$\exists A_2 \in R_2 \ \text{s.t.} \ |\phi\rangle = A_2 |\psi_{\text{singlet}}\rangle$$

For the moment let us continue to concentrate on the first part of the theorem, generating arbitrary states on $H_1 \otimes H_2$ by operations on H_1.

Insertion II

Thus denoting the projectors on H_1 resulting from the measurement of σ_{12} by $P^{\pm}$ and the 180° rotation of the spin' by Q, we are claiming that for any pure state $|\phi\rangle$ on $H_1 \otimes H_2$, $|\phi\rangle$ can be written in the form $A_1 |\psi_{\text{singlet}}\rangle$, where $A_1 = \alpha P_1^+ + \beta Q P_1^+ + \gamma P_1^- + \delta Q P_1^-$ for suitable choice of the complex coefficients α, β, γ and δ.

systems.
But ~~one~~ any state ~~of~~ for the joint system is some linear combination of these four states so by the mathematical operation of linear combination we can see how to generate an 'arbitrary' state in $H_1 \otimes H_2$ ~~by a combination of~~ from physical operations performed on particle no. 1 [illegible]
~~This is not Baby Reeh–Schlieder theorem.~~

The essential property of $|\Psi_{singlet}\rangle$ we have used in the argument is the mirror-image correlations. In terms of projector operators, writing $P_1^{\pm} = \frac{1}{2}(1 \pm \sigma_{1z})$ and $P_2^{\pm} = \frac{1}{2}(1 \pm \sigma_{2z})$ we are exploiting the fact that

$$Prob^{\Psi_{singlet}}\left(P_2^{\pm} = 1 \,\middle|\, P_1^{\mp} = 1\right) = 1 \qquad (2)$$

Clearly if we had any joint state in $H_1 \otimes H_2$ for which (2) was true, where $P_1^{\mp}$ are any pair of ~~orthogonal~~ distinct one-dimensional projectors in H_1 and $P_2^{\mp}$ some other pair of ~~orthogonal~~ distinct one-dimensional projectors in H_2 we could prove the first part of the Baby R-S theorem.

Putting the matter slightly differently a sufficient condition for the first part of the Baby R-S theorem to hold ~~is~~: in state $|\Psi\rangle$ is:

$$\forall P_2, \exists P_1 \text{ s.t. } Prob^{\Psi}(P_2 | P_1) = 1 \qquad (3)$$

where we have abbreviated $Prob^{\Psi}(P_2 = 1 | P_1 = 1)$ by $Prob^{\Psi}(P_2 | P_1)$

$P_1\psi/\|P_1\psi\|$

Now $\mathrm{Prob}^{\psi}(P_2/P_1) = \mathrm{Prob}(P_2 = 1)$

$$= \langle P_2 \rangle_{P_1\psi/\|P_1\psi\|}$$

$$= \langle P_1 P_2 \rangle_\psi / \langle P_1 \rangle_\psi \qquad (4)$$

where we have used the fact that P_1 and P_2 commute.

Eq. (4) is just the usual expression for a conditional probability as the ratio of a joint probability and a marginal.

So condition (3) can be written in the form

$$\forall P_2,\ \exists P_1 \text{ s.t. } \langle P_1 P_2 \rangle_\psi = \langle P_1 \rangle_\psi \qquad (3')$$

Let us now express (3') as a condition on the correlation coefficient $C(P_1, P_2)$ between P_1 and P_2.

We have

$$C(P_1, P_2) = \frac{\langle P_1 P_2 \rangle_\psi - \langle P_1 \rangle_\psi \langle P_2 \rangle_\psi}{\left[\langle P_1 \rangle_\psi (1 - \langle P_1 \rangle_\psi) \langle P_2 \rangle_\psi (1 - \langle P_2 \rangle_\psi)\right]^{1/2}}$$

So condition (3') becomes

$\forall P_2, \exists P_1$ s.t.

$$C(P_1, P_2) = \left(\frac{\langle P_1 \rangle_\psi (1 - \langle P_2 \rangle_\psi)}{\langle P_2 \rangle_\psi (1 - \langle P_1 \rangle_\psi)} \right)^{1/2} \qquad (3'')$$

~~Appendix~~ *

We begin by proving the following weaker result.

Theorem 5: The baby R–S theorem implies that

$\forall P_2, \exists P_1$ s.t.

$$\langle P_1 P_2\rangle_\psi \neq \langle P_1\rangle_\psi \langle P_2\rangle_\psi$$

We assume that P_2 is non-trivial, i.e. we exclude $P_2 = 0$ or 1 for which Theorem 5 clearly fails.

~~Section A (from p. 18)~~

Let $|\psi\rangle$ be a zero state for which the baby R–S theorem is true

proof: Assume $\langle P_1 P_2\rangle_\psi = \langle P_1\rangle_\psi \langle P_2\rangle_\psi$ for some given projector $P_2 \in R_2$ and for all projectors $P_1 \in R_1$.

Let $\hat{P}_2 = P_2 - \langle P_2\rangle_\psi I$

then $\langle P_1 \hat{P}_2\rangle_\psi = 0$, $\forall P_1 \in R_1$

So $\hat{P}_2|\psi\rangle$ is orthogonal to $P_1|\psi\rangle$ $\forall P_1 \in R_1$. But since any operator $A_1 \in R_1$ is a combination of projectors it follows that $\hat{P}_2|\psi\rangle$ is orthogonal to $A_1|\psi\rangle$ for $\forall A_1 \in R_1$. But from the Baby R–S theorem any vector in H is equal to $A_1|\psi\rangle$ for some A_1. So we conclude that $\hat{P}_2|\psi\rangle$ is orthogonal to every vector in H which implies $\hat{P}_2|\psi\rangle = 0$.

corollary to the theorem above, cited ~~#~~ ~~delete~~

But from the ~~Reeh–Schlieder~~ ~~theorem~~, this implies $\hat{P}_2 = 0$ i.e. $P_2 = \langle P_2\rangle_\psi I$, which is only possible if $P_2 = 0$ or I. So arguing contrapositively, for any non-trivial P_2, Theorem 5 is established.

We now show how to strengthen Theorem 5 to produce Theorem 4.

see back p. 16

Notice that (3'') does not say

$C(P_1, P_2) =$

$\forall P_2, \exists P_1$, s.t. $C(P_1, P_2) = 1$

This only applies when $\langle P_1 \rangle_\psi = \langle P_2 \rangle_\psi$,

a condition which is satisfied in the singlet state example but is by no means necessary in order to prove the baby R-S theorem.

So far we have exhibited (3') as a ___sufficient___ condition for deriving the first part baby R-S theorem.
We now want to demonstrate

Theorem 4: Condition (3') is a ___necessary___ condition for proving the baby R-S theorem.

In other words from the baby R-S theorem as an assumption we can ~~shall now~~ prove condition (3').

* ~~Insertion~~

Let $|\psi\rangle$ be some state for which the baby R-S theorem is true.
Denote by R_1 the von algebra of operators on H_1 and R_2 the algebra of operators on H_2. Because H_1 and H_2 are finite-dimensional R_1 and R_2 are trivially von Neumann algebras

Insertion A

~~Then~~ The baby R-S theorem says:

$\forall |\phi\rangle \in H_1 \otimes H_2, \exists A_1 \in R_1$ s.t. $|\phi\rangle = A_1 |\psi\rangle$

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II from back of previous page.

The proof is very straightforward.

For any $A_2 \in R_2$, if $A_2 |\psi\rangle = 0$, then

if $|\phi\rangle$ is any vector in $H_1 \otimes H_2$

$\exists A_1$, s.t. $|\phi\rangle = A_1 |\psi\rangle$, so

$A_2 |\phi\rangle = A_2 A_1 |\psi\rangle = A_1 A_2 |\psi\rangle = 0$.

Since $|\phi\rangle$ is an arbitrary vector,

it follows that $A_2 = 0$

choose $|\phi\rangle = P_2|\psi\rangle \big/ \| P_2|\psi\rangle \|$ (4)

Then by construction

$$\langle P_2 \rangle_\phi = 1 \qquad (5)$$

~~Den~~ Denote by C an ~~the~~ operator on H_1 for which

$$|\phi\rangle = C|\psi\rangle \qquad (6)$$

the existence of such a C is guaranteed by the ~~Gisin~~ Gisin R-S theorem.

~~It follows that from (6) and 5~~

Substituting (6) in (5) gives.

$$\langle \psi | Q P_2 | \psi \rangle = 1 \qquad (7)$$

where $Q = C^\dagger C$

Since Q is a positive Hermitian operator on H_1 we can expand.

$$Q = \lambda_1 \Pi_1 + \lambda_1' \Pi_1' \qquad (8)$$

~~From 7~~ where λ_1, λ_1' are the positive real eigenvalues of Q, and Π_1, Π_1' are orthogonal projections on H_1.

Substituting (8) in (7) yields.

$$w_1 \frac{\langle \Pi_1 P_2 \rangle_\psi}{\langle \Pi_1 \rangle_\psi} + w_2 \frac{\langle \Pi_1' P_2 \rangle_\psi}{\langle \Pi_1' \rangle_\psi} = 1 \qquad (9)$$

where $w_1 = \lambda_1 \langle \Gamma_1 \rangle_4$

$w_2 = \lambda_2 \langle \Gamma_1' \rangle_4$

But from (4) $\| |\phi\rangle \| = 1$

Hence from (6) $\| C|\psi\rangle \| = 1$

which implies $\langle \psi | Q | \psi \rangle = 1$

i.e. $\lambda_1 \langle \Gamma_1 \rangle_4 + \lambda_1' \langle \Gamma_1' \rangle_4 = 1$

or $w_1 + w_2 = 1$ . . — — . (10)

where $w_1 \geq 0$ and $w_2 \geq 0$.

~~From~~ Now, using (10), ~~the L.H.S~~

$$L.H.S. (9) \leq \text{Max}\left(\frac{\langle \Gamma_1 P_2 \rangle_4}{\langle \Gamma_1 \rangle_4}, \frac{\langle \Gamma_1' P_2 \rangle_4}{\langle \Gamma_1' \rangle_4} \right)$$

So in order to satisfy of (9) we require

$$\text{Max}\left(\frac{\langle \Gamma_1 P_2 \rangle_4}{\langle \Gamma_1 \rangle_4}, \frac{\langle \Gamma_1' P_2 \rangle_4}{\langle \Gamma_1' \rangle_4} \right) = 1$$

— — (11)

That is to say, one or other (or both) of Γ_1 and Γ_1' satisfy $(3')$ so by 'existential' generalization $(3')^8$ is true, Q.E.D.

What Theorem 4 shows is that from the baby R-S theorem we can deduce

that given any projector on $\mathcal{H}_2$ there always exists a projector on $\mathcal{H}_1$ which is not only correlated with it, but is maximally correlated, subject to fixed values of $\langle P_1 \rangle_4$ and $\langle P_2 \rangle_4$, i.e. achieves the value given in (3'').

Now ~~all~~ these conclusions generalize nicely to the field theory case, if we remember that the Reeh–Schlieder theorem asserts, not that any state in $\mathcal{H}$ can be generated from the vacuum, but only that any ~~a~~ state can be approximated as closely as we like (in norm) by acting on the vacuum with elements of $R(O)$.

So Theorem 4 becomes:

Theorem 4': For any two spacelike separated bounded open regions O_1 and O_2 and $\forall \varepsilon > 0$, $\forall P_2 \in R(O_2)$

$\exists P_1 \in R(O_1)$ s.t.

$$\langle P_1 P_2 \rangle_\Omega \geq (1-\varepsilon) \langle P_1 \rangle_\Omega$$

Footnote

We leave it as an exercise to the interested reader to provide the necessary epsilonics to prove Theorem 4'. (It follows as an simple ~~special~~ easy consequence ~~case~~ of a general result proved as Theorem 4 in Licht's (9) 1966 paper. But Licht does not seem to be aware of this corollary or its implications).

~~Inventors~~* The formal proof is sketched in the Appendix.

Appendix. Proof of Theorem 4'

Proof: Choose $\phi = \dfrac{P_2\Omega}{\|P_2\Omega\|}$

so by construction $\langle P_2\rangle_\phi = 1$, ~~and $\|\phi\|$~~

Then, by the Reeh-Schlieder theorem,
$\forall \varepsilon > 0$, $\exists\ C_1 \in R(O_1)$ s.t. $\|\phi' - \phi\| < \varepsilon$

where $\phi' = C_1\Omega$

As a preliminary lemma we ~~first~~ remark that C_1 can additionally be chosen so as to make $\|\phi'\| = 1$.

To see this, introduce $\phi'' = \phi' / \|\phi'\|$

so by construction $\|\phi''\| = 1$

~~Then $\|\phi'' - \phi''\|$~~ $\|\phi'' - \phi\|$

Then, from $\|\phi' - \phi\| < \varepsilon$, we can deduce.

$$\|\phi'' - \phi\| < \varepsilon' = \frac{\varepsilon\sqrt{2(1+\varepsilon)}}{1-\varepsilon}$$

So reverting to ϕ' in place of ϕ'' and ε in place of ε', the lemma is proved.

~~We~~ ~~next~~ notice next that

$$\langle P_2\rangle_{\phi'} > 1 - 2\varepsilon$$

This follows at once from the inequality

$$\left|\langle P_2\rangle_{\phi'} - \langle P_2\rangle_\phi\right| \le \|\phi'\|\cdot\|\phi' - \phi\| + \|\phi' - \phi\|\cdot\|\phi\|$$

Now ~~consider~~ write

$$\langle P_2 \rangle_{\phi'} = \langle Q_1 P_2 \rangle_\Omega$$

where $Q_1 = C_1^* C_1$ is bounded, self-adjoint and positive. Q_1 may be approximated arbitrarily closely by a finite sum of its spectral projections? What this means is that $\forall \epsilon' > 0$ we can choose ~~a~~ operator Q_1' such that $Q_1' = \sum_{i=1}^{n} \lambda_i \Gamma_1^i$

where $\lambda_i \geq 0$, Γ_1^i are ~~projection~~ projectors in $R(O_1)$, n is a finite integer and

$\forall \epsilon' > 0$, $\| Q_1' - Q_1 \| < \epsilon'$.

In general $\langle Q_1' \rangle_\Omega \neq 1$, but as in our previous lemma we can always adjust Q_1', simply by dividing it by $\langle Q_1' \rangle_\Omega$, so that the additional condition $\langle Q_1' \rangle_\Omega = 1$ is satisfied.

This means that we can always arrange that

$$\sum_{i=1}^{n} \lambda_i \langle \Gamma_1^i \rangle_\Omega = 1$$

Now consider $\langle Q_1' P_2 \rangle_\Omega$

Since $\left| \langle Q_1' P_2 \rangle_\Omega - \langle Q_1 P_2 \rangle_\Omega \right|$

$$= \left| \langle (Q_1' - Q_1) P_2 \rangle_\Omega \right| < \epsilon'$$

it follows that $\langle Q_1' P_2 \rangle_\Omega > \langle Q_1 P_2 \rangle_\Omega - \epsilon'$

$$= \langle P_2 \rangle_{\phi'} - \epsilon' > 1 - 2\epsilon - \epsilon'$$

But
$$\langle \theta_1' \rho_2 \rangle_\Omega = \sum_{i=1}^{n} w_i \frac{\langle \Gamma_1^i \rho_2 \rangle_\Omega}{\langle \Gamma_1^i \rangle_\Omega}$$

where $w_i = \lambda_i \langle \Gamma_1^i \rangle_\Omega$

and hence $\sum_{i=1}^{n} w_i = 1$

So $\langle \theta_1' \rho_2 \rangle_\Omega \leq \mathrm{Max} \left\{ \frac{\langle \Gamma_1^i \rho_2 \rangle_\Omega}{\langle \Gamma_1^i \rangle_\Omega} \right\}$

~~But each quantity in this set is ≤ 1~~

Thus ~~So~~ it follows that
$$\mathrm{Max} \left\{ \frac{\langle \Gamma_1^i \rho_2 \rangle_\Omega}{\langle \Gamma_1^i \rangle_\Omega} \right\} > 1 - 2\varepsilon - \varepsilon'$$

or replacing $2\varepsilon + \varepsilon'$ by ε we obtain finally that one or more of $\langle \Gamma_1^i \rho_2 \rangle_\Omega / \langle \Gamma_1^i \rangle_\Omega$ is greater than $1 - \varepsilon$, from which ~~the~~ Theorem 4' follows immediately

3 ~~Conclusions~~ On the distance-dependence of the Vacuum Correlations

It is important to realize that vacuum correlations are not independent of distance, as in Bell-type correlations, but fall off exponentially with distance on a scale set by the Compton wavelength of a massive field, or the detector wavelength of a photon field.[10,11] ~~Thus it is well known that vacuum correlations maximally violate the Bell inequality~~[11,12,13] ~~i.e. achieving the so-called Cirelson bound of $2\sqrt{2}$ against 2 as compared with the classical limit of 2 for the Bell inequalities. But the violation falls off exponentially with distance so making a source-free variant of the Bell experiment impossible from the practical point of view.~~

Do these results conflict with our claim that independent of distance maximal correlations always exist?

In order to investigate this question, we consider the Fredenhagen bound on the correlations.

Applied to projectors $P_1 \in R(O_1)$ and $P_2 \in R(O_2)$ Fredenhagen's theorem[10] says:

$$\langle P_1 P_2 \rangle_\Omega - \langle P_1 \rangle_\Omega \cdot \langle P_2 \rangle_\Omega$$

$$\cancel{\leq e^{-mr} \sqrt{\|P_1 \Omega\|^2 \cdot \|P_2 \Omega\|^2}}$$

$$\leq e^{-ml} \cancel{\leq e^{-mr}} \|P_1 \Omega\| \cdot \|P_2 \Omega\| \qquad \text{-- (12)}$$

where m is the mass-gap between the vacuum and the lowest excited state

(minimum) lengthy distance between   space-like separation

l is the ~~spatial~~ separation of the regions O_1 and O_2 (~~as assuming units which~~ make $\hbar = c = 1$)

From (12) we obtain immediately the following bound on the correlation coefficient:

$$c(P_1, P_2) \leq e^{-ml} \cdot \frac{1}{\sqrt{(1-\langle P_1\rangle_\Omega)\cdot(1-\langle P_2\rangle_\Omega)}} \qquad (13)$$

Comparing (13) with (3'') we see that consistency of ~~theorem~~ our these two results requires

$$\langle P_1\rangle_\Omega \leq \frac{e^{-2ml}\,\langle P_2\rangle_\Omega}{(1-\langle P_2\rangle_\Omega)^2} \quad --- (14)$$

In other words the P_1 whose existence is asserted in (3'') must also satisfy (14) as a result of Fredenhagen's theorem.
So, for ~~for given $\langle P_2\rangle_\Omega$ the maximally correlated P_1~~ a given probability of P_2 happening, the ~~probability~~ maximally correlated P_1 must have a probability of occurring that falls off exponentially with the distance between O_1 and O_2.
This again shows how difficult it would be in practice to observe the long range correlations in the vacuum. But of course it does not show that they don't exist!

X[~~How might such small probabilities for the maximally correlated P_1 seem to occur?~~

P.T.O.

These ~~results~~ considerations are ~~clearly~~ closely related to the well-known results of Landau[12,13], showing that the vacuum correlations maximally violate the Bell inequalities.
Landau[13] chooses three arbitrary spacelike separated regions and shows that local projections can always be chosen in these three regions so as to produce a maximal violation of the Bell inequality. Our own analysis is not directly addressed to the question of 'classical' reconstruction of RQFT along hidden-variable lines but seeks to discuss the correlations themselves.

As a fairly heuristic remark one notices that in any local algebra $R(O)$ one can always find a sequence of mutually space-like separated regions $O^1, O^2, O^3 \ldots$ such that $P = P^1 \cdot P^2 \cdot P^3 \ldots \cdot P^N$ is a member of $R(O)$, while $P^1 \in R(O^1)$, $P^2 \in R(O^2)$ etc. If we assume statistical independence of P^1, P^2 etc., then since $\forall i \; \langle P^i \rangle_\Omega < 1$, it follows that $\langle P \rangle_\Omega$ will get smaller and smaller as ~~more and more~~. N is made larger and larger. In other words ~~a~~ possible candidate for a projector with a small probability of occurring is ~~one~~ ~~joint meas~~ which is associated with a joint measurement of a sequence of projectors in disjoint space-like separated regions, which are statistically independent.

* ~~Some of these excitations exhibit~~ particle-like properties

4 Conclusion What is being detected by a local measurement in the vacuum?
We have argued that it is not a particle but a local ~~field~~ observable in the sense made precise in algebraic quantum field theory.
But there are other answers to this question in the literature, which we want to discuss briefly.
In ^chapter VI of^ his 1992 book Local Physics Haag[5] describes the approach of himself and his collaborators. N-particle states are ones in which N-field correlations, but no higher order, exist in the field. However Haag admits that ~~these~~ particle states he is characterizing are not strictly local, but spread with non-vanishing amplitude over unbounded regions. So from the point of view of local physics in Haag's sense strictly speaking particles are never observable — they are an idealization which lead to a plethora of misunderstandings about what is going on in quantum field theory. The theory is about fields and their local excitations *. That is all there is to it.

Insertion

* , Jeremy Butterfield, Guido Bacciagaluppi
, and Thomas Breuer

who in particular ~~who~~ suggested the proof of Theorem 5

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discussions on these issues.
The paper is dedicated to Jean-Pierre Vigier,
whose insights into ~~about~~ the nature of the
quantum vacuum have inspired his students
and colleagues alike.

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To prove $\forall P_2\ \exists P_1$ s.t. $\langle P_2 P_1\rangle_\Omega > (1-\varepsilon)\ \langle P_1\rangle_\Omega$

1. Note $|\langle\phi|\psi\rangle| \leq \| |\phi\rangle\| \cdot \|\psi\|$.

Proof ~~$|\langle\phi|\psi\rangle|$~~ $(\langle\phi| - \langle\psi|)(|\phi\rangle - |\psi\rangle) \geq 0$

See Simmons p. 246

$\therefore \langle\phi|\phi\rangle + \langle\psi|\psi\rangle - 2\,\mathrm{Re}\langle\phi|\psi\rangle \geq 0$

$\therefore \mathrm{Re}\langle\phi|\psi\rangle \leq \frac{1}{2}(\langle\phi|\phi\rangle + \langle\psi|\psi\rangle)$

Now $\left(\sqrt{\langle\phi|\phi\rangle} - \sqrt{\langle\psi|\psi\rangle}\right)^2 \geq 0$

$\therefore \langle\phi|\phi\rangle + \langle\psi|\psi\rangle - 2\|\phi\|\cdot\|\psi\| \geq 0$

$\therefore \|\phi\|\cdot\|\psi\| \leq \frac{1}{2}(\langle\phi|\phi\rangle + \langle\psi|\psi\rangle)$

This is equivalent to $\langle\phi|\psi\rangle\langle\psi|\phi\rangle \leq \langle\phi|\phi\rangle\cdot\langle\psi|\psi\rangle$

2. Note $\|x+y\| \leq \|x\| + \|y\|$ ✓

Proof. $\|x-y\| \geq \big|\|x\| - \|y\|\big|$ ✓

So $\big|\|x\| - \|y\|\big| \leq \|x+y\| \leq \|x\| + \|y\|$

Also we have. $\|Tx\| \leq \|T\|\cdot\|x\|$.

and. $\|T^*\| = \|T\|$

Consider $\langle P_2 \rangle_\psi = 1$ where $\psi = \frac{P_2 \Omega}{\|P_2 \Omega\|}$ $\quad \|\psi\| = 1$

now take ψ' where $\|\psi'\| = 1$ and $\psi' = C_1 \Omega$

and $\|\psi' - \psi\| \leq \varepsilon_1$

then $\langle P_2 \rangle_{\psi'} = \langle \psi' | P_2 | \psi' \rangle \leq 1$

since $P_2 + (1 - P_2) = 1$

and $\langle P_2 \rangle_{\psi'} \geq 1 - \varepsilon'$ ——— (1)

since. $|\langle P_2 \rangle_{\psi'} - \langle P_2 \rangle_\psi|$

$= |\langle \psi' | P_2 | \psi' \rangle - \langle \psi | P_2 | \psi \rangle|$

$= |\langle \psi' | P_2 | \psi' \rangle - \langle \psi' | P_2 | \psi \rangle + \langle \psi' | P_2 | \psi \rangle - \langle \psi | P_2 | \psi \rangle|$

$= |\langle \psi' | P_2 | \Delta\psi \rangle + \langle \Delta\psi | P_2 | \psi \rangle|$

where $\Delta\psi = \psi' - \psi$.

$\leq |\langle \psi' | P_2 | \Delta\psi \rangle| + |\langle \Delta\psi | P_2 | \psi \rangle|$

$\leq \underbrace{\|\psi'\|}_{\leq 1} \cdot \|\Delta\psi\| + \|\Delta\psi\| \cdot \underbrace{\|\psi\|}_{\leq 1}$

$= 2 \|\Delta\psi\| = 2\varepsilon_1$

So choose. $\varepsilon' = 2\varepsilon_1$

Then (1) follows.

Now consider $\langle P_2 \rangle_{\psi'} = \langle Q P_2 \rangle_\psi$ $\quad Q = C^+ C$

and compare $\langle Q' P_2 \rangle_\psi$ where $Q' = \sum \lambda_i P_i$

$Q = \int_0^{\|Q\|} \lambda \, dP(\lambda)$

then $\|Q' - Q\| < \varepsilon_2$

So $|\langle Q\rho_2\rangle_\psi - \langle Q'\rho_2\rangle_\psi|$

$= |\langle\psi|Q\rho_2|\psi\rangle - \langle\psi|Q'\rho_2|\psi\rangle|$

$= |\langle\psi|(Q'-Q)\rho_2|\psi\rangle|$

$\le \underbrace{\|\psi\|}_{=1} \cdot \varepsilon_2 \underbrace{\|\psi\|}_{=1} = \varepsilon_2.$

$\therefore \langle Q'\rho_2\rangle_\psi \geq \langle Q\rho_2\rangle_\psi - \varepsilon_2.$

$= \langle\rho_2\rangle_{\psi'} - \varepsilon_2$

$\geq 1 - 2\varepsilon_1 - \varepsilon_2$

$= 1 - \varepsilon$, where $\varepsilon = 2\varepsilon_1 + \varepsilon_2.$

But $\langle Q'\rho_2\rangle_\psi = \sum w_i \dfrac{\langle\rho_i\rho_2\rangle_\psi}{\langle\rho_i\rangle_\psi}.$

where $w_i = \lambda_i \langle\rho_i\rangle_\psi$

$\therefore \sum_i w_i = \langle Q'\rangle_\psi \le 1 + \varepsilon_2.$

$\le \sum w_i \max\left(\dfrac{\langle\rho_i\rho_2\rangle_\psi}{\langle\rho_i\rangle_\psi}\right)$

$\le (1+\varepsilon_2) \max\left(\dfrac{\langle\rho_i\rho_2\rangle_\psi}{\langle\rho_i\rangle_\psi}\right)$

∴ we require for consistency writing $x = \max \dfrac{\langle\rho_i\rho_2\rangle_\psi}{\langle\rho_i\rangle_\psi}$

$(1+\varepsilon_2)x \geq 1 - 2\varepsilon_1 - \varepsilon_2.$

$\therefore x \geq \dfrac{1-2\varepsilon_1-\varepsilon_2}{1+\varepsilon_2} \geq (1-2\varepsilon_1-\varepsilon_2)(1-\varepsilon_2)$

$= 1 - 2\varepsilon_1 - 2\varepsilon_2 + 2\varepsilon_1\varepsilon_2 + \varepsilon_2^2$

$= 1 - \varepsilon$

where $\varepsilon = 2\varepsilon_1 + 2\varepsilon_2 - 2\varepsilon_1\varepsilon_2 - \varepsilon_2^2$

Suppose we find. $\|\psi' - \cancel{\psi}\| \leq \varepsilon$ (1)

where $\|\psi'\| \neq 1$

Then replace ψ' by $\psi'' = \psi' / \|\psi'\|$

and we have. $\|\psi''\| = 1$ by construction

<u>and</u> $\|\psi'' - \psi\| \geq \big| \|\psi''\| - \|\psi\| \big| = 0.$

But $\|\psi'' - \psi\| = \left\| \frac{\psi'}{\|\psi'\|} - \psi \right\|$

and from (1) $\big| \|\psi'\| - \|\psi\| \big| \leq \varepsilon$

$\therefore \big| \|\psi'\| - 1 \big| \leq \varepsilon.$

so $\|\psi'\| \leq 1 + \varepsilon$

and $\geq 1 - \varepsilon.$

$\therefore \psi'' = \frac{\psi'}{1 \pm \varepsilon.}$

and $\cancel{\|\psi'' - \psi''\|}$ $\|\psi'' - \psi\| = \left\| \frac{\psi' - q\psi}{q} \right\|$

where $q = 1 \pm \varepsilon.$, $= \frac{1}{q} \|\psi' - q\psi\|$

$= \frac{1}{q} \sqrt{(\psi' - q\psi, \psi' - q\psi)} = \frac{1}{q} \sqrt{\psi'^2 + q^2 \psi^2 - q^{*}\left((\psi, \psi') + \psi'\psi\right)}$

$= \frac{1}{q} \sqrt{\psi'^2 + \psi^2 - (\psi, \psi') - (\psi', \psi) + (q^2 - 1)[\psi^2] - (q-1)[(\psi, \psi') + (\psi', \psi)]}$

$\leq \frac{1}{q} \sqrt{\varepsilon^2 + (q-1)\cancel{(\varepsilon + 1)}(q+1)\left(\|\psi\|^2 - 2\,\mathrm{Re}(\psi, \psi')\right)} \sim \frac{\sqrt{\varepsilon}}{1 \pm \varepsilon}$

~~$\leq \varepsilon$, where $\varepsilon = \frac{1}{q}\sqrt{\varepsilon^2 + (q-1)(q+1)\left(\|\psi\|^2 - 2\,\mathrm{Re}(\psi, \psi')\right)}$~~

$\therefore \|\psi'' - \psi\|$

$$\leq \frac{1}{q}\sqrt{\varepsilon^2 + (q-1)\left[(q+1) - 2\,\mathrm{Re}(\psi,\psi')\right]}$$

$$\leq \frac{1}{1-\varepsilon}\sqrt{\varepsilon^2 + (q-1)\left[(q+1) - 2\,\mathrm{Re}(\psi,\psi')\right]}$$

where $q > 1-\varepsilon$ and $q < 1+\varepsilon$

$\therefore q-1 > -\varepsilon$ and $q-1 < \varepsilon$

$\therefore |q-1| < \varepsilon$

$q = \|\psi'\|$

and $\|\psi' - \psi\| \leq \varepsilon$ and $\|\psi\| = 1$

whence $(\psi' - \psi, \psi' - \psi) \leq \varepsilon^2$

or $\psi'^2 + \psi^2 - 2\,\mathrm{Re}(\psi,\psi') \leq \varepsilon^2$

or $q^2 + 1 - 2\,\mathrm{Re}(\psi,\psi') \leq \varepsilon^2$

$\therefore q^2 - q + \left[(q+1) - 2\,\mathrm{Re}(\psi,\psi')\right] \leq \varepsilon^2$

or $\left[(q+1) - 2\,\mathrm{Re}(\psi,\psi')\right] \leq \varepsilon^2 + q(1-q)$

now $\cancel{1+\varepsilon} - q \leq \varepsilon - 1$

$\therefore 1 - q \leq \varepsilon$, and $1-q \geq -\varepsilon$

$\therefore |1-q| \leq \varepsilon$

$\therefore$

$\leq \varepsilon^2 + (1+\varepsilon)\varepsilon$

$= \varepsilon + 2\varepsilon^2$

we also know that $\|\psi' - \psi\|^2 = q^2 + 1 - 2\,\mathrm{Re}(\psi,\psi') \geq 0$

So $(q+1) - 2\,\mathrm{Re}(\psi,\psi') \geq q(1-q)$

$\geq (1-\varepsilon)(-\varepsilon) = -\varepsilon(1-\varepsilon)$

$= -\varepsilon + \varepsilon^2$

$\therefore \left| (q+1) - 2\,\mathrm{Re}(\psi, \psi') \right|$

$\leq \max\left(\varepsilon + 2\varepsilon^2, \ \varepsilon - \varepsilon^2\right)$

$= \varepsilon + 2\varepsilon^2$

$\therefore \|\psi'' - \psi^{\#}\|$

$\leq \frac{1}{1-\varepsilon} \sqrt{\varepsilon^2 + |q-1| \cdot |(q+1) - 2\,\mathrm{Re}(\psi, \psi')|}$

$= \frac{1}{1-\varepsilon} \sqrt{\varepsilon^2 + \varepsilon(\varepsilon + 2\varepsilon^2)}$

$= \frac{1}{1-\varepsilon} \sqrt{2\varepsilon^2(1+\varepsilon)}$

$= \frac{\varepsilon}{1-\varepsilon} \sqrt{2(1+\varepsilon)} = \varepsilon''$ say

where $\varepsilon'' = \frac{\varepsilon\sqrt{2(1+\varepsilon)}}{1-\varepsilon}$

can be made as small as we like by making ε sufficiently small

or write $Q'' = \dfrac{Q'}{\langle Q' \rangle_{\psi}}$

then by construction $\langle Q'' \rangle_{\psi} = 1$

So we recover $\sum_i w_i' = 1$

where $w_i' = \dfrac{w_i}{\langle Q' \rangle_{\psi}}$.

and then, given $\| Q' - Q \| \leq \varepsilon_2$

we can show

$\| Q'' - Q \| \leq \varepsilon_3$

$\left(= \left\| \dfrac{Q'}{q} - Q \right\| \right.$ $\qquad q = \langle Q' \rangle_{\psi} = 1 \pm \varepsilon$

$= \left\| \dfrac{Q' - qQ}{q} \right\| = \dfrac{1}{q} \| Q' - qQ \|$

and then seem similar analysis as for ψ''.

Q'' and ψ'' are the operators & vectors used by London (1987) — give [illegible] [illegible] proof as compared with ψ' and Q' with $\varepsilon = 2\varepsilon_1 + \varepsilon_2$

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Thcs $\rightarrow$ i

$|a-b| \leq \varepsilon_2$

$a > b$ $a-b \leq \varepsilon_2$

$a \leq b + \varepsilon_2$

or $b \geq a - \varepsilon_2$

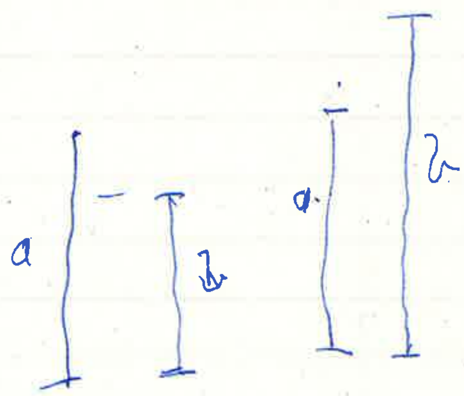
$\cancel{a >}$

$b \geq a - \varepsilon_2.$

$a < b.$ $b - a \leq \varepsilon_2$

$b \leq a + \varepsilon_2$

$b > a > a - \varepsilon_2.$



$b \geq a - \varepsilon_2.$

and. $b \leq a + \varepsilon_2$

$\sqrt{a-b}$

$\leq$

$a < \varepsilon$

Fredenhagen bound says

$$\langle P_1 P_2\rangle_\psi - \langle P_1\rangle_\psi \langle P_2\rangle_\psi$$

$$\leq e^{-mr}\sqrt{\|P_1\psi\|^2 \cdot \|P_2\psi\|^2}$$

$$= e^{-mr} \cdot \|P_1\psi\| \cdot \|P_2\psi\|$$

$$= e^{-mr}\sqrt{\langle P_1\rangle_\psi \cdot \langle P_2\rangle_\psi}$$

$$\therefore\ c(P_1,P_2) \leq e^{-mr} \frac{1}{\sqrt{(1-\langle P_1\rangle_\psi)(1-\langle P_2\rangle_\psi)}}$$

cf Redhead saturation bound for $c(P_1,P_2)$

$$= \sqrt{}$$

For correlation we require:

$$\sqrt{\frac{\langle P_1\rangle_\psi \cdot (1-\langle P_2\rangle_\psi)}{(1-\langle P_1\rangle_\psi)\cdot\langle P_2\rangle_\psi}} \leq e^{-mr}\frac{1}{\sqrt{(1-\langle P_1\rangle_\psi)(1-\langle P_2\rangle_\psi)}}$$

ie.
$$\langle P_1\rangle_\psi \leq \frac{e^{-2mr}\cdot\langle P_2\rangle_\psi}{(1-\langle P_2\rangle_\psi)^2}.$$

$$= \left(\frac{\sqrt{\langle P_1 \rangle_\psi} \cdot \sqrt{1 - \langle P_2 \rangle_\rho}}{(1 - \langle P_1 \rangle_\psi) \langle P_2 \rangle_\psi} \right)^{1/2}$$

under the conditions of Theorem 3'.

For fixed $\langle P_2 \rangle_\psi$ & $\langle P_1 \rangle_\psi$ this is the maximum possible value for the correlation coefficient. It becomes equal to one only if $\langle P_2 \rangle_\rho = \langle P_1 \rangle_\rho$.

As we shall see later in order to select a ΣP_1 satisfying Theorem 3' we require in general $\langle P_2 \rangle_\rho \ll \langle P_1 \rangle_\rho \ll 1$

Under these conditions the maximum possible value for $C(P_2, P_1)$ is extended approximately as $\sqrt{\frac{\langle P_1 \rangle_\rho}{\langle P_2 \rangle_\rho}}$, so the large conditional probability $\mathrm{Prob}(P_1 | P_2)$ is arrived at consistently with a low value of the correlation coefficient. It is important to realise that it is the large value of the conditional probability that is important for our argument, not a large value for the correlation coefficient.